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Global Solutions to a Model of Structural Phase Transitions in Shape Memory Alloys

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Global in time existence and uniqueness of solutions to a nonlinear system of coupled evolution equations is discussed. The system represents a model of dynamical structural phase transitions (martensitic transformations) in shape memory alloys. © 1988 Academic Press, Inc.

1. INTRODUCTION

In this paper, we consider a nonlinear boundary value problem for the following system of differential equations

$$u_{tt} - \left[\frac{\partial}{\partial \varepsilon}\psi(\Theta, \varepsilon)\right]_x - \mu u_{xxt} = f(x, t), \qquad (1.1a)$$

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$$-\Theta\left[\frac{\partial}{\partial\Theta}\psi(\Theta,\varepsilon)\right]_t-k\Theta_{xx}-\alpha k\Theta_{xxt}-\mu u_{xt}^2=\lambda(x,t), \tag{1.1b}$$

$$\varepsilon=u_x, \tag{1.1c}$$

to be satisfied in $\Omega=(0,1)\subset\mathbb{R}$ for $t>0$, subject to a given function $\psi=\psi(\Theta,\varepsilon)$ which is nonconvex over some range of (Θ,ε). The system is to be complemented by the boundary conditions:

$$u=0, \tag{1.1d}$$

$$k\Theta_\nu=k_1(\Theta_\Gamma-\Theta), \tag{1.1e}$$

on $\Gamma=\{0,1\}$ for $t>0$, where $\Theta_\nu=-\Theta_x$ at $x=0$ and $\Theta_\nu=\Theta_x$ at $x=1$, and by the initial conditions

$$u(x,0)=u_0(x),\qquad u_t(x,0)=u_1(x)\qquad\text{for}\quad x\in\bar{\Omega}, \tag{1.1f}$$

$$\Theta(x,0)=\Theta_0(x)\qquad\text{for}\quad x\in\bar{\Omega}, \tag{1.1g}$$

where k,α,k_1,μ are positive constants, and $f(x,t)$, $\lambda(x,t)$, $\Theta_\Gamma(t)$, $u_0(x)$, $u_1(x)$, $\Theta_0(x)$ represent some given functions.

Problem (1.1) arises from modeling dynamical phase transitions in shape memory alloys (cf., Section 2). u stands for the displacement and Θ represents the absolute temperature. The terms μu_{xxt} ($\alpha k\Theta_{xxt}$, respectively) indicate the presence of viscosity (short thermal memory, respectively) in the material. We prove global in time existence and uniqueness of a weak solution to problem (1.1). These results extend those of [5, 8] where due to slightly different structural assumptions only local in time existence and uniqueness could be proved.

We should point out at this place that our structural assumptions are—from the physical viewpoint—not more restrictive than those imposed in [5, 8]. Indeed, the only changes made in comparison to [5, 8] apply for very large and small temperatures and for large strains. Thus, within the range where the shape memory effect occurs, we have the same model. On the other hand, our assumptions guarantee that the partial differential equation which represents the energy balance cannot change its character from parabolic forwards to backwards in time (which possibility was not excluded in [5, 8]). From this viewpoint, the asymptotic assumptions made in this paper even appear to be more reasonable.

Finally, let us note that if either viscosity or thermal memory (or both) are not present in the model, i.e., if $\mu=0$ or $\alpha=0$, the presented technique does not work. (1.1) appears to be a rather difficult open problem in this case.

2. Some Phenomenology

Structural phase transitions in solids give rise to a large number of qualitatively new mathematical problems. We are going to deal with a problem which resulted from the mathematical modeling of the dynamics of martensitic transformations in shape memory alloys.

Martensitic transformations represent diffusionless solid state phase transitions, connected with a deformation of the crystal lattice which produces a macroscopic strain (cf., [3]). As an additional phenomenon accompanying such transitions, a *shape memory effect* (sometimes also referred to as pseudoelastic behavior) is observed for a considerable number of materials, in particular, for various metallic alloys. Shape memory manifests in an alternative stability of either low-symmetric or highly-symmetric equilibrium crystal structures of the alloy. A unique place among shape memory alloys, due to extraordinary thermomechanical properties, occupies the family of Ti–Ni alloys, commonly classified as Nitinol (cf., [3, 6]). Nitinol may deform of more than 50% strain prior to fracture; it also exhibits an enormous shape memory, being capable of recovering more than 8% strain by heating (cf., [6]). This results in its numerous applications in heat engines (e.g.), offering pollution-free, low-cost energy conversion units (cf., [3]).

In our considerations, we follow the construction of a macroscopic continuum model for the phase transitions in shape memory alloys, developed in [2, 8] for Nitinol-like materials. The one-dimensional model has the form of the nonlinear coupled system (1.1), complemented by relevant constitutive laws which correspond to:

(i) introduction of the free energy ψ in the Landau–Devonshire form,

(ii) taking dissipation effects into account,

(iii) postulating heat conduction with short (fading) memory.

As far as the free energy is concerned, substantial information is comprehended in its strongly temperature-dependent qualitative form,

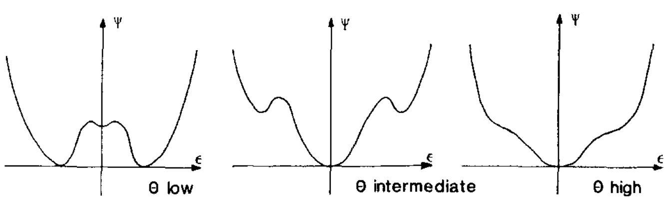
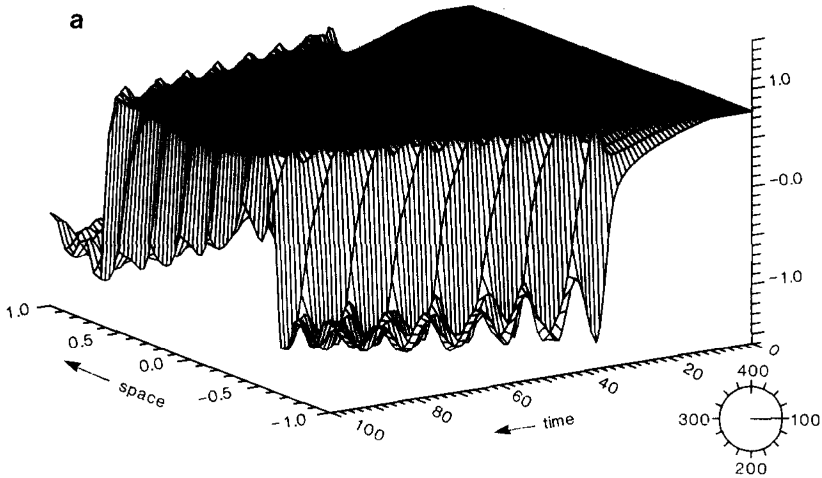


Fig. 1. Isothermal equilibrium specific free energy $\Psi = \Psi(\Theta, \varepsilon)$ as function of strain ε in the autonomous situation (no external loads) for different temperature ranges.

a



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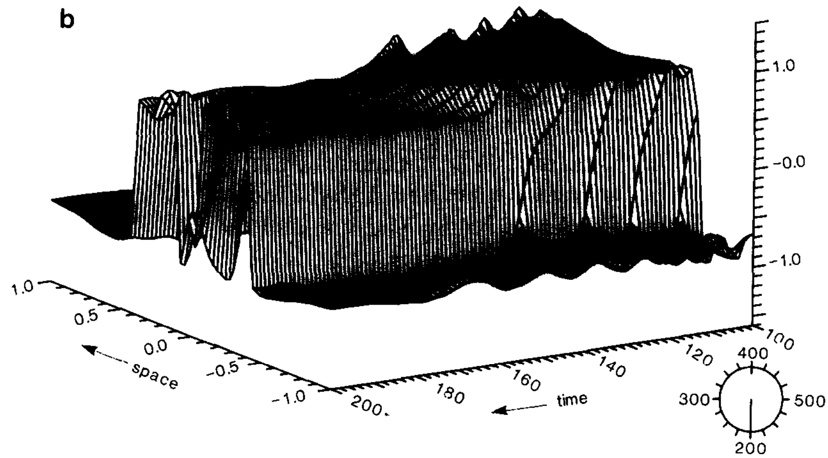

FIG. 2. Deformation: (a) $\Theta = 1.00$, $\sigma = 0.00$, $t \in [0, 100]$; (b) $\Theta = 1.00$, $\sigma = 0.00$, $t \in [100, 200]$; (c) $\Theta = 1.80$, $\sigma = 0.10$, $t \in [400, 500]$; (d) $\Theta = 1.80$, $\sigma = 0.10$, $t \in [500, 600]$.

reflecting the shape memory. Schematically depicted in Fig. 1 in the autonomous situation (no external mechanical loads), there are two lateral minima of the free energy $\psi = \psi(\Theta, \varepsilon)$ with respect to the strain ε at low values of the temperature Θ; these minima are then dominated by an additional minimum arising at the origin as temperature exceeds some critical value, and eventually only the central minimum exists for temperatures above the Curie point (cf., [1, 3–7]).

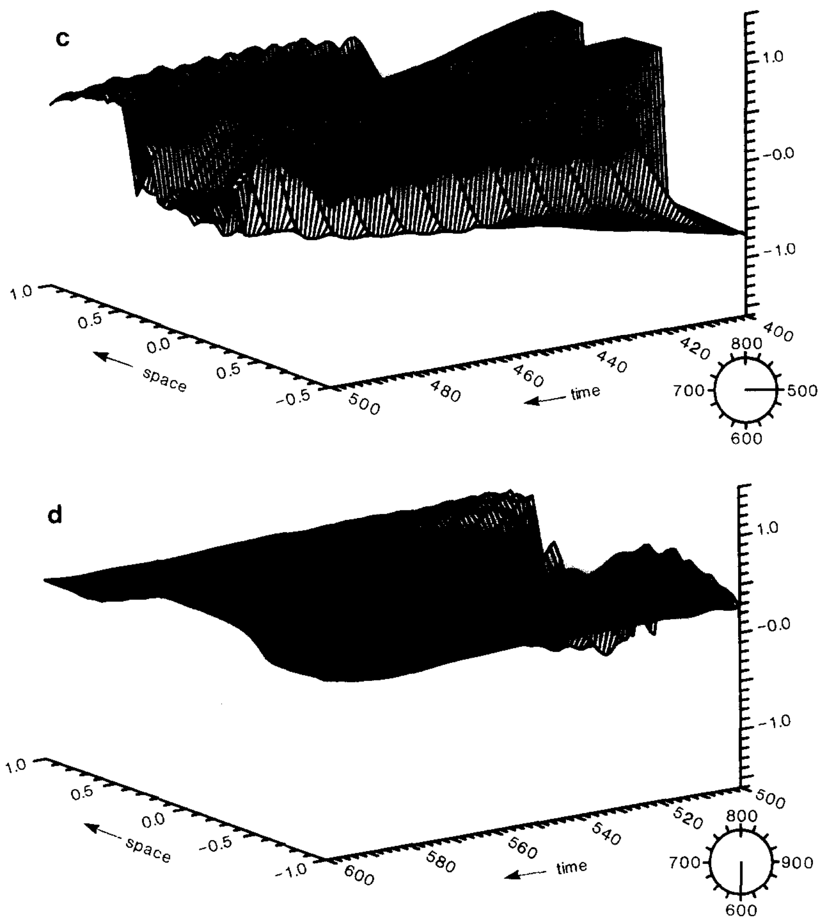

FIG. 2—*Continued.*

In the macroscopic model developed in [2, 8], balance equations for momentum and energy are assumed to govern the dynamics of processes in shape memory alloys. As it is indicated by results of the numerical simulation (cf., [2]), these equations actually produce the experimentally observed behavior. In particular, Figs. 2a–d show the development of the process under some stronger hypotheses in as much as both viscosity and short thermal memory terms are not present which gives rise to the creation and propagation of sharp interfaces between the different phases. Figs. 2a–b correspond to the situation when a load exceeding the yield limit has been applied without any accompanying external thermal action (low temperature range throughout); the propagation of the interfaces up to their stabilization and also nucleation phenomena may be seen there. Only the two twin phases related to the lateral minima of ψ (cf., Fig. 1) are stable. The situation in Figs. 2c–d refers to the process subject to simultaneous strong heating and small external loading applied after the stabilization of the development depicted in Figs. 2a–b. The shape memory effects are now visible, resulting in a stabilization of the phase which corresponds to the central minimum of ψ (for a detailed exposition, see [2]).

These numerical results provide encouragement for a further exploration of the model, with the viscosity ($\mu > 0$) and short thermal memory ($\alpha > 0$) components contributing to a smoother behavior of the solutions and no longer sharply divided phases. Thus, in mathematical terms, one can treat those terms as just playing a regularizing role.

3. Structural Hypotheses

Following the construction in [8], we postulate the specific free energy ψ to be a function of temperature Θ and strain ε alone. To obtain the desired shape memory effect, we assume ψ in the Landau–Devonshire form

$$\psi(\Theta, \varepsilon) = \psi_0(\Theta) + \psi_1(\Theta)\,\varepsilon^2 + \psi_2(\Theta, \varepsilon), \tag{3.1}$$

with the term $\psi_0(\Theta)$ representing pure heat conduction, $\psi_1(\Theta)\,\varepsilon^2$ providing shape memory, and $\psi_2(\Theta, \varepsilon)$ characterizing nonlinear temperature-dependent elasticity.

Note that (3.1) is the simplest possible decomposition which is capable of producing all the above-mentioned effects. To be more specific, let us qualitatively depict the functions $\psi_i(\Theta)$ in Fig. 3 (also cf., [6]), where we admit

$$\psi_2(\Theta, \varepsilon) = -a\varepsilon^4 + b\varepsilon^6 \qquad \text{for} \quad |\varepsilon| < \varepsilon_M \tag{3.2a}$$

$$\psi_2(\Theta, \varepsilon) = \psi_3(\Theta)\,\varepsilon^2 \qquad \text{for} \quad |\varepsilon| > 2\varepsilon_M \tag{3.2b}$$

with some finite positive constants ε_M, a, b.

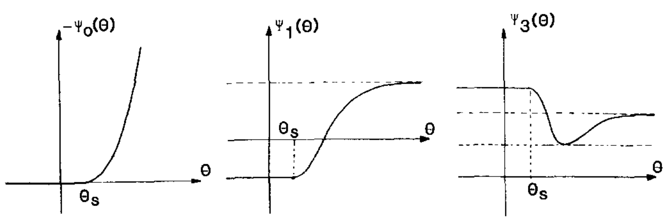

Figure 3

The proposed approximation differs from that of [8] in the term $\psi_2(\Theta, \varepsilon)$ which is here assumed temperature-dependent in order to reflect the thermodynamic correctness of the model over arbitrary time intervals better.

We impose the following structural hypotheses on the model:

(H1) $\psi \in C^3(\mathbb{R} \times \mathbb{R})$,

(H2) $(\partial/\partial\Theta)\, \psi(\Theta, \varepsilon) = 0$ for $\Theta \leqslant \Theta_S$, $\varepsilon \in \mathbb{R}$, as well as

(H3) $-\Theta(\partial^2/\partial\Theta^2)\, \psi(\Theta, \varepsilon) \geqslant 0$,

(H4) $|\Theta^2(\partial^2/\partial\Theta\, \partial\varepsilon)\, \psi(\Theta, \varepsilon)| \leqslant C|\varepsilon|$, $|(\partial^2/\partial\Theta\, \partial\varepsilon)\, \psi(\Theta, \varepsilon)| \leqslant C|\varepsilon|$,

(H5) $|(\partial^2/\partial\varepsilon^2)\, \psi(\Theta, \varepsilon)| \leqslant C$, $|(\partial/\partial\varepsilon)\, \psi(\Theta, \varepsilon)| \leqslant C|\varepsilon|$, for all $(\Theta, \varepsilon) \in [\Theta_S, \infty) \times \mathbb{R}$, where $\Theta_S \geqslant 0$ is given (cf., [8]);

(H6) $|\int_{\Theta_S}^{\infty} \Theta^2(\partial^3/\partial\Theta^2\, \partial\varepsilon)\, \psi(\Theta, \varepsilon)\, d\Theta| \leqslant C|\varepsilon|$ for all $\varepsilon \in \mathbb{R}$, with a positive constant C.

The above hypotheses (H1)–(H6) may be formulated individually for the components ψ_i of the free energy. For $\psi_0(\Theta)$ and $\psi_1(\Theta)$, one can just impose the same hypotheses as in [8], with a physical explanation given there. As far as $\psi_2(\Theta, \varepsilon)$ is concerned, as additional hypotheses on $\psi_3(\Theta)$ we have to impose (cf., (3.2b)):

(i) $\psi_3 \in C^3(\mathbb{R})$.

(ii) There exist positive constants ψ_{3m}, ψ_{3M} such that $0 < \psi_{3m} \leqslant \psi_3(\Theta) \leqslant \psi_{3M}$ for all $\Theta \in \mathbb{R}$,

(iii) $\psi_3(\Theta) = \text{const}$ for $\Theta \leqslant \Theta_S$,

(iv) $|\Theta^2 \psi_3'(\Theta)| \leqslant C$, $\Theta \in \mathbb{R}$,

(v) $\int_{\Theta_S}^{\infty} \Theta^2 |\psi_3''(\Theta)|\, d\Theta \leqslant C < +\infty$, and of special significance, to guarantee (H3),

(vi) $\psi_3''(\Theta) + \psi_1''(\Theta) \leqslant 0$ in some neighborhood of Θ_S.

The last hypothesis is substantial. It modifies the situation discussed in [8] in that now (H3) is globally assured. Due to this hypothesis, in the energy balance equation (1.1b) the coefficient of Θ_t cannot become negative; consequently, the local in time solution of [8] now turns out to be global.

Finally, let us remark that in the following proof the assumption

$$\left|\frac{\partial^2}{\partial\Theta\, \partial\varepsilon}\, \psi(\Theta, \varepsilon)\right| \leqslant C|\varepsilon|, \qquad (\Theta, \varepsilon) \in [\Theta_S, \infty) \times \mathbb{R},$$

is superfluous if $\Theta_S > 0$.

4. Global Existence and Uniqueness of Solutions

In this section, we formulate and prove a global existence result for system (1.1), subject to the free energy having the form (3.1).

In addition to the structural hypotheses (H1)–(H6) imposed on $\psi(\Theta, \varepsilon)$, we assume the following for the data of problem (1.1):

(H7) $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$; $(\Omega = (0, 1) \subset \mathbb{R}^1)$

(H8) $\Theta_0 \in H^2(\Omega)$, $\Theta_0(x) > \Theta_S$ for $x \in \bar{\Omega}$;

(H9) $\lambda \in L^2(0, T; L^2(\Omega))$ for any $T > 0$, $\lambda \geqslant 0$;

(H10) $f \in L^2(0, T; L^2(\Omega))$ for any $T > 0$,

(H11) $\Theta_\Gamma \in H^1(0, T)$ for any $T > 0$,

$$\Theta'_\Gamma(t) \geqslant 0, \; \Theta_\Gamma(t) > \Theta_S \qquad \text{for} \quad t \geqslant 0.$$

Recall that $\Theta_S \geqslant 0$ is the constant as in structural hypotheses (H1)–(H6). In the sequel, we shall use the notations

$$\Omega_t = \Omega \times (0, t), \qquad \|\cdot\|\text{-norm in } L^2(\Omega), \qquad \|\Theta(t)\|_\Gamma^2 = \Theta^2(0, t) + \Theta^2(1, t).$$

The main result of the paper is given by the following.

Theorem. *Under hypotheses* (H1)–(H11), *problem* (1.1), (3.1), (3.2) *admits a unique global solution* (Θ, u) *such that for any* $T > 0$:

(i) $u \in H^2(0, T; L^2(\Omega)) \cap H^1(0, T; H_0^1(\Omega) \cap H^2(\Omega))$,

(ii) $u_{xt} \in C([0, T]; L^2(\Omega))$,

(iii) $\Theta \in H^1(0, T; H^2(\Omega))$.

Before proving this theorem, let us observe the following.

Remark 1. The local existence of solutions to problem (1.1), (3.1), (3.2) has been proved in [8] by using a special Galerkin approximation. A proof of the uniqueness of that solution has been given in [5]. The assumptions on the data postulated in [8] were slightly more restrictive than those we are going to work with.

Proof of the theorem. To begin with, let us notice that (1.1a) is actually a nonlinear viscoelastoplasticity equation in terms of u, while Eq. (1.1b) is pseudoparabolic with respect to Θ. Both are strongly coupled by the terms containing derivatives of the free energy ψ.

Let us formulate the following auxiliary linearized problem in terms of (Θ, u):

$$u_{tt} - \mu u_{xxt} = f(x, t) + \left[\frac{\partial}{\partial \varepsilon} \psi(\bar{\Theta}, \bar{\varepsilon}) \right]_x, \tag{4.1a}$$

$$-\Theta \frac{\partial}{\partial \Theta^2} \psi(\bar{\Theta}, \bar{\varepsilon})\, \Theta_t - k\Theta_{xx} - \alpha k \Theta_{xxt} = \lambda(x, t) + \mu \bar{\varepsilon}_t^2 + \frac{\partial^2}{\partial \Theta\, \partial \varepsilon} \psi(\bar{\Theta}, \bar{\varepsilon})\, \bar{\varepsilon}_t, \tag{4.1b}$$

$$\bar{\varepsilon} = \bar{u}_x, \tag{4.1c}$$

with $(\bar{\Theta}, \bar{u})$ fixed. Then the corresponding nonlinear resolution operator $(\bar{\Theta}, \bar{u}) \mapsto (\Theta, u)$, acting in the space characterized by assertions (i)–(iii) of the theorem, is contractive on a sufficiently small time interval $[0, T_0]$, and hence admits a unique fixed point (Θ, u) which represents the unique local solution of problem (1.1), (3.1), (3.2). Due to hypothesis (H8), we may claim that $\Theta(x, t) \geqslant \Theta_S$, on $\bar{\Omega} \times [0, T_0]$. Since the arguments are standard and exploit just the a priori bounds we shall establish in the sequel, we omit details.

Once we have got the local existence and uniqueness of the solution on $[0, T_0]$, to prove the relevant global results on any $[0, T]$, $T > T_0$, we need to derive some uniform a priori estimates. This will be accomplished in the subsequent lemmas.

Remark 2. Throughout the paper, C, C_i will represent uniform constants which may only depend on T and the data, but are independent of T_0.

LEMMA 1. *For all* $t \in [0, T_0]$,

$$\|u_x(t)\|^2 + \|u_t(t)\|^2 + \int_0^t \|u_{xt}(s)\|^2\, ds \leqslant C. \tag{4.2}$$

Proof. Multiply Eq. (1.1a) by u_t and integrate over Ω_t, to obtain

$$\frac{1}{2}\left[\|u_x(t)\|^2 + \|u_t(t)\|^2\right] + \mu \int_0^t \|u_{xt}(s)\|^2\, ds$$
$$= \int_{\Omega_t}\!\!\int f u_t + \frac{1}{2}\left[\|u_0'\|^2 + \|u_1\|^2\right] + \int_{\Omega_t}\!\!\int \left(u_x - \frac{\partial \psi}{\partial \varepsilon}\right) u_{xt}. \tag{4.3}$$

Due to hypothesis (H5), the right-hand side is bounded by

$$\frac{1}{2}\left(\|u_0'\|^2 + \|u_1\|^2\right) + \frac{\mu}{2} \int_0^t \|u_{xt}(s)\|^2\, ds$$
$$+ \frac{C_1}{\mu} \int_0^t \|u_x(s)\|^2\, ds + \frac{1}{2} \int_0^t \|u_t(s)\|^2\, ds + \frac{1}{2} \int_0^t \|f\|^2\, ds.$$

Hence, by Gronwall's inequality, (4.2) follows. ∎

LEMMA 2. *For all* $t \in [0, T_0]$,

$$\|\Theta_x(t)\|^2 + \int_0^t \|\Theta_x(s)\|^2\, ds + \|\Theta(t)\|_\Gamma^2 + \int_0^t \|\Theta(s)\|_\Gamma^2\, ds \leqslant C, \tag{4.4}$$

$$\sup_{(x,t)\in \bar{\Omega}_{T_0}} |\Theta(x,t)| \leqslant C. \tag{4.5}$$

Proof. Multiply Eq. (1.1b) by Θ and integrate over Ω_t, to get

$$\int_{\Omega_t}\int \left[-\Theta^2 \left(\frac{\partial}{\partial \Theta} \psi(\Theta, \varepsilon) \right)_t - k\Theta\Theta_{xx} - \alpha k \Theta\Theta_{xxt} - \lambda\Theta - u_{xt}^2 \Theta \right] = 0. \tag{4.6}$$

Consider the individual terms in (4.6):

$$\begin{aligned}
-\int_{\Omega_t}\int (k\Theta\Theta_{xx} + \alpha k \Theta\Theta_{xxt}) &= k\int_0^t \|\Theta_x(s)\|^2\, ds + \frac{\alpha k}{2}\|\Theta_x(t)\|^2 - \frac{\alpha k}{2}\|\Theta_0'\|^2 \\
&\quad + k_1 \int_0^t \int_\Gamma \Theta(\Theta - \Theta_\Gamma) + \frac{\alpha k_1}{2}\|\Theta(t)\|_\Gamma^2 \\
&\quad - \frac{\alpha k_1}{2}\|\Theta_0\|_\Gamma^2 - \alpha k_1 \int_\Gamma \Theta(t)\, \Theta_\Gamma'(t),
\end{aligned} \tag{4.7a}$$

$$\begin{aligned}
-\int_{\Omega_t}\int \Theta^2 \left[\frac{\partial}{\partial\Theta}\psi(\Theta,\varepsilon)\right]_t &= \int_{\Omega_t}\int \left(-\frac{\partial^2 \psi}{\partial \Theta^2}\Theta^2 \Theta_t - \Theta^2 \frac{\partial^2\psi}{\partial\Theta\,\partial\varepsilon}\varepsilon_t \right) \\
&= \int_{\Omega_t}\int \frac{\partial}{\partial t}\left[\int_{\Theta_S}^{\Theta(x,t)} \left(-v^2 \frac{\partial^2}{\partial\Theta^2}\psi(v,\varepsilon)\right) dv \right] \\
&\quad + \int_{\Omega_t}\int \left(\int_{\Theta_S}^{\Theta(x,t)} v^2 \frac{\partial^3}{\partial\Theta^2\,\partial\varepsilon}\psi(v,\varepsilon)\, dv\right)\varepsilon_t \\
&\quad - \int_{\Omega_t}\int \Theta^2 \frac{\partial^2\psi}{\partial\Theta\,\partial\varepsilon}\varepsilon_t \\
&= \int_\Omega \left(\int_{\Theta_S}^{\Theta(x,t)} \left(-v^2 \frac{\partial^2}{\partial\Theta^2}\psi(v,\varepsilon)\right) dv\right) \\
&\quad + \int_\Omega \left(\int_{\Theta_S}^{\Theta_0(x)} v^2 \frac{\partial^2}{\partial\Theta^2}\psi(v, u_0')\, dv\right) \\
&\quad + \int_{\Omega_t}\int \left(\int_{\Theta_S}^{\Theta(x,t)} v^2 \frac{\partial^3}{\partial\Theta^2\,\partial\varepsilon}\psi(v,\varepsilon)\, dv\right)\varepsilon_t \\
&\quad - \int_{\Omega_t}\int \Theta^2 \frac{\partial^2\psi}{\partial\Theta\,\partial\varepsilon}\varepsilon_t.
\end{aligned} \tag{4.7b}$$

In view of hypotheses (H4)–(H6), by Lemma 1 we get the bound

$$\left|\int_{\Omega_t}\int\left(\int_{\Theta_S}^{\Theta(x,t)} v^2\frac{\partial^3}{\partial\Theta^2\,\partial\varepsilon}\psi(v,\varepsilon)\,dv\right)\varepsilon_t\right| \leqslant C\int_{\Omega_t}\int|\varepsilon|\,|\varepsilon_t|$$
$$\leqslant\frac{C}{2}\int_0^t[\|u_x(s)\|^2+\|u_{xt}(s)\|^2]\,ds\leqslant C_3,$$

$$\left|\int_{\Omega_t}\int\Theta^2\frac{\partial^2\psi}{\partial\Theta\,\partial\varepsilon}\varepsilon_t\right|\leqslant C\int_{\Omega_t}\int|\varepsilon|\,|\varepsilon_t|\leqslant C_3. \tag{4.8a}$$

By using Poincaré's inequality, we can conclude

$$\|\Theta(t)\|^2\leqslant C[\|\Theta_x(t)\|^2+\|\Theta(t)\|_\Gamma^2],$$

hence, via Young's inequality with $\delta>0$,

$$\left|\int_{\Omega_t}\int\lambda\Theta\right|\leqslant\frac{C\delta}{2}\left\{\int_0^t\|\Theta_x(s)\|^2\,ds+\int_0^t\|\Theta(s)\|_\Gamma^2\,ds\right\}+\frac{1}{2\delta}\int_0^t\|\lambda(s)\|^2\,ds, \tag{4.8b}$$

$$\int_0^t\int_\Gamma\Theta\Theta_\Gamma\leqslant\frac{1}{2}\left[\int_0^t\|\Theta(s)\|_\Gamma^2\,ds+\int_0^t\|\Theta_\Gamma(s)\|_\Gamma^2\,ds\right], \tag{4.8c}$$

$$\left|\int_\Gamma\Theta(t)\,\Theta'_\Gamma(t)\right|\leqslant\frac{\delta}{2}\|\Theta(t)\|_\Gamma^2+\frac{1}{2\delta}\|\Theta'_\Gamma(t)\|_\Gamma^2. \tag{4.8d}$$

Therefore, taking δ appropriately small, we obtain from (4.6)

$$\|\Theta_x(t)\|^2+\int_0^t\|\Theta_x(s)\|^2\,ds+\int_0^t\|\Theta(s)\|_\Gamma^2\,ds+\|\Theta(t)\|_\Gamma^2$$
$$\leqslant C_4+C_5\left|\int_{\Omega_t}\int\Theta u_{xt}^2\right|\leqslant C_4+C_5\left\{\sup_{\Omega_t}|\Theta|\right\}\int_0^t\|u_{xt}(s)\|^2\,ds$$
$$\leqslant C_4+C_6\sup_{\Omega_t}|\Theta|. \tag{4.9}$$

Since

$$\sup_{\bar\Omega_t}|\Theta(x,t)|\leqslant C\sup_{\tau\in[0,t]}[\|\Theta_x(\tau)\|^2+\|\Theta(\tau)\|_\Gamma^2]^{1/2},$$

we can conclude from (4.9) that for any $t\in[0,T_0]$

$$\|\Theta_x(t)\|^2+\|\Theta(t)\|_\Gamma^2+\int_0^t\|\Theta_x(s)\|^2\,ds+\int_0^t\|\Theta(s)\|_\Gamma^2\,ds$$
$$\leqslant C_7+C_8\,\delta\sup_{\tau\in[0,t]}\{\|\Theta_x(\tau)\|^2+\|\Theta(\tau)\|_\Gamma^2\}. \tag{4.10}$$

By taking in (4.10) the upper bound with respect to $t \in [0, T_0]$, after adjusting δ small enough (in dependence solely on T), we arrive at (4.4), (4.5). ∎

LEMMA 3. *For any* $t \in [0, T_0]$,

$$\|u_{xt}(t)\|^2 + \|u_{xx}(t)\|^2 + \int_0^t \|u_{xxt}(s)\|^2\, ds \leqslant C, \tag{4.11}$$

$$\sup_{\Omega_t} |\varepsilon(x, t)| \leqslant C. \tag{4.12}$$

Proof. Let us multiply Eq. (1.1a) by $-u_{xxt}$ and then integrate over Ω_t. We get

$$-\int_{\Omega_t}\int \left[u_{tt} - \left(\frac{\partial\psi}{\partial\varepsilon}\right)_x - \mu u_{xxt} - f\right] u_{xxt} = 0, \tag{4.13a}$$

hence

$$\tfrac{1}{2}[\|u_{xt}(t)\|^2 + \|u_{xx}(t)\|^2] - \tfrac{1}{2}[\|u_1'\|^2 + \|u_0''\|^2] + \mu \int_0^t \|u_{xxt}(s)\|^2\, ds$$
$$= \int_{\Omega_t}\int \left(u_{xx} - \frac{\partial^2\psi}{\partial\varepsilon^2} u_{xx} - \frac{\partial^2\psi}{\partial\Theta\,\partial\varepsilon}\Theta_x\right) u_{xxt} + \int_{\Omega_t}\int f u_{xxt}. \tag{4.13b}$$

Due to hypotheses (H2), (H4), (H5), by Young's inequality,

$$\left|\int_{\Omega_t}\int u_{xx}u_{xxt}\right| \leqslant \frac{\delta}{2}\int_0^t \|u_{xxt}(s)\|^2\, ds + \frac{1}{2\delta}\int_0^t \|u_{xx}(s)\|^2\, ds,$$
$$\left|\int_{\Omega_t}\int \frac{\partial^2\psi}{\partial\varepsilon^2} u_{xx}u_{xxt}\right| \leqslant \frac{C\,\delta}{2}\int_0^t \|u_{xxt}(s)\|^2\, ds + \frac{C}{2\delta}\int_0^t \|u_{xx}(s)\|^2\, ds, \tag{4.14a}$$

$$\left|\int_{\Omega_t}\int \frac{\partial^2\psi}{\partial\Theta\,\partial\varepsilon}\Theta_x u_{xxt}\right| \leqslant C\int_{\Omega_t}\int |\varepsilon\Theta_x u_{xxt}|$$
$$\leqslant \left\{C \sup_{\tau\in[0,t]} \|\Theta_x(\tau)\|\right\} \int_0^t \|\varepsilon(s)\|_{L^\infty(\Omega)} \|u_{xxt}(s)\|\, ds$$
$$\leqslant C\left\{\sup_{\tau\in[0,t]} \|\Theta_x(\tau)\|\right\}\left\{\frac{\delta}{2}\int_0^t \|u_{xxt}(s)\|^2\, ds\right.$$
$$\left. + \frac{1}{2\delta}\int_0^t \|\varepsilon(s)\|^2_{L^\infty(\Omega)}\, ds\right\}. \tag{4.14b}$$

Because $u|_\Gamma = 0$, for each $t \in [0, T_0]$ there is some $x_0(t) \in (0, 1)$, such that $u_x(x_0(t), t) = 0$. Thus

$$u_x(x, t) = \int_{x_0(t)}^{x} u_{xx}(\xi, t)\, d\xi,$$

and

$$|u_x(x, t)|^2 \leqslant |x - x_0(t)| \left| \int_{x_0(t)}^{x} u_{xx}^2(\xi, t)\, d\xi \right| \leqslant \|u_{xx}(t)\|^2. \tag{4.15}$$

Due to (4.15) and (4.5), it follows from (4.14b) that

$$\left| \int_{\Omega_t}\int \frac{\partial^2 \psi}{\partial \Theta\, \partial \varepsilon}\, \Theta_x u_{xxt} \right| \leqslant C_9\, \delta \int_0^t \|u_{xxt}(s)\|^2\, ds + C_{10} \int_0^t \|u_{xx}(s)\|^2\, ds. \tag{4.16}$$

Furthermore, by Young's inequality,

$$\left| \int_{\Omega_t}\int f u_{xxt} \right| \leqslant \frac{\delta}{2} \int_0^t \|u_{xxt}(s)\|^2\, ds + \frac{1}{2\delta} \int_0^t \|f(s)\|^2\, ds. \tag{4.17}$$

Take δ sufficiently small in (4.17), to conclude eventually from (4.13b) that

$$\|u_{xt}(t)\|^2 + \|u_{xx}(t)\|^2 + \int_0^t \|u_{xxt}(s)\|^2\, ds \leqslant C_{11} + C_{12} \int_0^t \|u_{xx}(s)\|^2\, ds.$$

Hence, by Gronwall's inequality, we can conclude (4.11) and (4.12). ∎

LEMMA 4. *For all* $t \in [0, T_0]$,

$$\int_0^t \|u_{tt}(s)\|^2\, ds \leqslant C. \tag{4.18}$$

Proof. (4.18) is a direct consequence of Lemmas 1–3 applied to Eq. (1.1a). ∎

LEMMA 5. *For all* $t \in [0, T_0]$,

$$\int_0^t \|\Theta_{xt}(s)\|^2\, ds + \int_0^t \|\Theta_t(s)\|_\Gamma^2\, ds \leqslant C, \tag{4.19}$$

$$\int_0^t \|\Theta_t(s)\|^2\, ds \leqslant C. \tag{4.20}$$

Proof. Multiply Eq. (1.1b) by Θ_t and then integrate over Ω_t, to obtain

$$\int_{\Omega_t}\int \left\{-\Theta\left[\frac{\partial}{\partial\Theta}\psi(\Theta,\varepsilon)\right]_t - k\Theta_{xx} - \alpha k\Theta_{xxt} - \mu u_{xt}^2 - \lambda\right\}\Theta_t = 0 \tag{4.21a}$$

and, consequently,

$$\begin{aligned}&\frac{k}{2}\|\Theta_x(t)\|^2 - \frac{k}{2}\|\Theta_0'\|^2 + \alpha k\int_0^t \|\Theta_{xt}(s)\|^2\,ds + k_1\int_0^t\int_\Gamma (\Theta-\Theta_\Gamma)\,\Theta_t\\ &\quad+\alpha k_1\int_0^t\int_\Gamma \Theta_t(\Theta_t-\Theta_\Gamma') - \mu\int_{\Omega_t}\int u_{xt}^2\Theta_t\\ &\quad-\int_{\Omega_t}\int \lambda\Theta_t - \int_{\Omega_t}\int \Theta\Theta_t\left[\frac{\partial}{\partial\Theta}\psi(\Theta,\varepsilon)\right]_t = 0.\end{aligned} \tag{4.21b}$$

Because, due to hypotheses (H3) and (H4),

$$\begin{aligned}-\int_{\Omega_t}\int \Theta\Theta_t\left[\frac{\partial}{\partial\Theta}\psi(\Theta,\varepsilon)\right]_t &= -\int_{\Omega_t}\int \Theta\Theta_t^2\frac{\partial^2\psi}{\partial\Theta^2} - \int_{\Omega_t}\int \Theta\Theta_t u_{xt}\frac{\partial^2\psi}{\partial\Theta\,\partial\varepsilon}\\ &\geqslant -C\int_{\Omega_t}\int |\Theta_t\Theta u_{xt}u_x|,\end{aligned}$$

equality (4.21b) implies that

$$\begin{aligned}&\|\Theta_x(t)\|^2 + \int_0^t \|\Theta_{xt}(s)\|^2\,ds + \int_0^t \|\Theta_t(s)\|_\Gamma^2\,ds + \|\Theta(t)\|_\Gamma^2\\ &\quad\leqslant C_{13} + C_{14}\left\{\int_{\Omega_t}\int |\Theta_t|\,|\varepsilon_t|^2 + \int_{\Omega_t}\int |\Theta_t\Theta u_{xt}u_x|\right\}.\end{aligned} \tag{4.22a}$$

In view of Lemmas 1–3, by Young's and Poincaré's inequalities,

$$\begin{aligned}\int_{\Omega_t}\int |\Theta_t|\,|u_{xt}|^2 &\leqslant \int_0^t \|\Theta_t(s)\|_{L^\infty(\Omega)}\|u_{xt}(s)\|^2\,ds\\ &\leqslant \left\{\sup_{\tau\in[0,t]}\|u_{xt}(\tau)\|\right\}\left\{\frac{\delta}{2}\int_0^t \|\Theta_t(s)\|_{L^\infty(\Omega)}^2\,ds\right.\\ &\quad\left.+\frac{1}{2\delta}\int_0^t \|u_{xt}(s)\|^2\,ds\right\}\\ &\leqslant C_{15} + C_{16}\,\delta\int_0^t \{\|\Theta_{xt}(s)\|^2 + \|\Theta_t(s)\|_\Gamma^2\}\,ds,\end{aligned} \tag{4.22b}$$

$$\int_{\Omega_t}\int |\Theta\Theta_t u_{xt} u_x| \leqslant \left\{\sup_{\bar{\Omega}_t} |\Theta|\right\}\left\{\sup_{\bar{\Omega}_t} |\varepsilon|\right\}\int_{\Omega_t}\int |\Theta_t u_{xt}|$$

$$\leqslant C\left\{\frac{\delta}{2}\int_0^t \|\Theta_t(s)\|^2\,ds + \frac{1}{2\delta}\int_0^t \|u_{xt}(s)\|^2\,ds\right\}$$

$$\leqslant C_{17} + C_{18}\,\delta\left\{\int_0^t \|\Theta_{xt}(s)\|^2\,ds + \int_0^t \|\Theta_t(s)\|^2_{\Gamma}\,ds\right\}. \qquad (4.22c)$$

By taking $\delta > 0$ sufficiently small, we can conclude (4.19) and (4.20) directly from (4.22a)–(4.22c). ∎

LEMMA 6. *For all* $t \in [0, T_0]$,

$$\|\Theta_{xx}(t)\|^2 + \int_0^t \|\Theta_{xx}(s)\|^2\,ds \leqslant C, \qquad (4.23)$$

$$\int_0^t \|\Theta_{xxt}(s)\|^2\,ds \leqslant C. \qquad (4.24)$$

Proof. Multiply Eq. (1.1b) by $-\Theta_{xx}$ and then integrate over Ω_t:

$$k\int_0^t \|\Theta_{xx}(s)\|^2\,ds + \frac{\alpha k}{2}\|\Theta_{xx}(t)\|^2 - \frac{\alpha k}{2}\|\Theta_0''\|^2$$

$$+\int_{\Omega_t}\int \Theta\left[\frac{\partial}{\partial\Theta}\psi(\Theta,\varepsilon)\right]_t \Theta_{xx} + \mu\int_{\Omega_t}\int u_{xt}^2\Theta_{xx} + \int_{\Omega_t}\int \lambda\Theta_{xx} = 0. \qquad (4.25)$$

Recall that $u_t|_\Gamma = 0$ in view of the boundary condition (1.1d). Therefore, as in (4.15), we have

$$\int_0^t \|u_{xt}(s)\|^2_{L^\infty(\Omega)}\,ds \leqslant \int_0^t \|u_{xxt}(s)\|^2\,ds, \qquad (4.26a)$$

$$\left|\int_{\Omega_t}\int u_{xt}^2\Theta_{xx}\right| \leqslant \int_0^t \|u_{xt}(s)\|_{L^\infty(\Omega)}\,\|u_{xt}(s)\|\,\|\Theta_{xx}(s)\|\,ds$$

$$\leqslant \left\{\sup_{\tau\in[0,t]} \|u_{xt}(\tau)\|\right\}\left\{\frac{\delta}{2}\int_0^t \|\Theta_{xx}(s)\|^2\,ds + \frac{1}{2\delta}\int_0^t \|u_{xt}(s)\|^2_{L^\infty(\Omega)}\,ds\right\}$$

$$\leqslant C\left\{\frac{\delta}{2}\int_0^t \|\Theta_{xx}(s)\|^2\,ds + \frac{1}{2\delta}\int_0^t \|u_{xxt}(s)\|^2\,ds\right\}. \qquad (4.26b)$$

Taking $\delta > 0$ sufficiently small in (4.26b) and following the arguments used in the proof of Lemma 5, we conclude the estimate (4.23).

The estimate (4.24) is a direct consequence of (4.23) and Lemmas 1–5. ∎

All estimates in Lemmas 1–6 are uniform with respect to t. More precisely, none of the constants in those estimates is dependent on T_0. Therefore, by combining the local existence and uniqueness with the a priori estimates given by Lemmas 1–6, we can assert to have proved the relevant global results. The proof of the theorem is complete. ∎

Remark 3. The solution (Θ, u) of problem (1.1), (3.1), (3.2) is a classical one, provided the data and free energy are sufficiently regular.

Remark 4. Following the arguments of [8], we can claim that $\Theta(x, t) \geqslant \Theta_S$, globally.

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